Some Math You May Have Missed

Last week, 500 of America's top young mathletes took part in the USA Math Olympiads (USAMO and JrMO). You may wonder how they were selected. It began in November with the American Math Contests (there is a contest for 10^{th} grade or younger students that leads to the Jr. Olympiad, and a contest for 12^{th} grade or younger that leads to the senior contest). Out of over 100,000 contestants, about 5,000 qualify for the American Invitational Math Exam (AIME), which leads to the Olympiad.

I believe that all students, even those without ambition to go on to the next round, would benefit from the problem solving challenge of the AMC (or other math contests). Unfortunately, though the contests claim to only require high school level math, without calculus, in reality a typical high school curriculum is not adequate preparation to do one's best. Some topics are typically covered only superficially, and others not at all. I'd like to talk about a few of those; I think the ideas are interesting, and the problems you can solve with them rewarding, regardless of if you are interested in mathematics competitions.

Combinatorics

I consider combinatorics as "fancy ways to count things that are hard to count." Most students learn some ideas, such as the number of ways to pick a subset of k elements from a set of size n. The answer is $nCk = \binom{n}{k} = \frac{n!}{k!(n-k)!}$ if the set is unordered, or $nPk = \binom{n}{k}k!$ if the order of choosing the subset matters. But exercises rarely ask the student to do more than decide which formula is appropriate; critical thinking about the principles behind them is too often ignored.

When do you multiply [when you are making consecutive independent choices], when do you add [when you've separated into disjoint cases], when do you subtract or divide to correct for 'double counting', and when is it appropriate to divide for computing probabilities? I will illustrate these ideas with a sample problem.

[2024 AMC8 #25] A small airplane has 4 rows of 3 seats each. Eight passengers board the plane and choose their seats at random. Next, a couple boards together, what is the probability they find two adjacent empty seats?

The first 8 passengers have $\binom{12}{8} = \frac{12*11*10*9}{4*3*2*1} = 11*5*9$ ways to select their seats, so that is the denominator of our probability ratio [notice that I do not multiply out the factors, since that will not make my final calculation any easier.]
The numerator is the number of ways that the couple may find two adjacent empty seats. I will break the ways into disjoint cases. In case 1 it is possible that they find an entire empty row – here there are 4 possible rows and then 9 possibilities for the single remaining empty seat. A total of $4*9$ ways.
In the second case, there is no empty row yet they find a row with two empty adjacent seats.

Here there are 4 possible rows, 2 choices for the occupied edge seat in that row, and $\binom{9}{2} = 9 * \frac{8}{2}$ possibilities for the other two empty seats. But $4 * 2 * 36$ over counts because the remaining empty seats may also be adjacent. We exclude the double counting by subtracting the number of ways to have two pairs of adjacent empty seats. This is $\binom{4}{2} = 6$ ways to choose the rows, and two ways to choose the filled edge seat in each row.
Finally, the probability is: $\frac{4*9+4*2*9*4-6*2*2}{11*5*9} = \frac{100}{11*5*3} = \frac{20}{33}$.

A fun thing about counting problems is that there are often different approaches that will work. A frustrating thing is that it is sometimes hard to find an error in an argument. Think about this seemingly simpler solution, which gives an incorrect result. Can you find the flaws?

Suppose the couple is able to sit together – there are 4 rows, and 2 choices per row for them to sit, with $\binom{10}{8}$ ways for the remaining 8 passengers to fill in the rest. This gives $4 * 2 * \binom{10}{8}$ successful seatings. On the other hand, the total number of ways the passengers could sit is $\binom{12}{10}$ ways to choose which seats are filled, with $\binom{10}{2}$ ways to choose which of those 10 seats have the couple. So the probability is $\frac{4*2*\binom{10}{8}}{\binom{12}{10}\binom{10}{2}} = \frac{4*2}{6*11} = \frac{4}{33}$ [notice that the fraction reduces because $\binom{10}{8} = \binom{10}{2}$]. But this is only 1/5 of the correct probability. This does correctly count the things it claims to be counting, but it is not a correct way to find the requested probability. One error is that this calculation overcounts the sample space by including situations where the couple is sitting apart yet one of the couple is sitting next to an empty seat, and there are other mistakes as well.

Number Theory

To academics, Number Theory is sometimes called "The Queen of Mathematics". It is the branch of math that deals with the properties and patterns of the whole numbers, with the prime numbers playing a central role. It was considered the furthest possible from applied mathematics, until computers and cryptography came around.

Most students learn some facts about prime numbers, also things like least common multiple and greatest common factor, and perhaps divisibility tests (at least for 2, 3, 5, and 9). But this education is often superficial, and usually ends when they start learning algebra. For example, we learn how to use 'clock math' to determine that four and a half hours after 10:45 is 3:15, but it is rarely tied in with a greater understanding of modular arithmetic. The range of topics is quite broad, so I will only illustrate with a few examples of problems and solutions.

[2017 AMC 12B #19] Consider the 79-digit number $N =$123456789101112…4344, which is formed by writing the integers 1 through 44 down in order. What is the remainder when N is divided by 45?

Since 9 is a factor of 45, we will check the sum of the digits of N. $1 + 2 + \cdots + 9 = 45$ and there are four copies of this set of digits from the 'units digits.' The sum of all of these is a multiple of 9, so we can ignore it in our sum. Then from the tens digits there are 10 copies of 1, 2, and 3, and 5 copies of 4 which adds up to a total of 80. Finally, a 0, 1, 2, 3, and 4 from the unit's digits in the 40’s makes the sum 90. We see that N is a multiple of 9. Furthermore, subtract 9 from N and the units digit will be 5, which means $N - 9$ is a multiple of 5 as well as still being a multiple of 9. Thus, $N - 9$ is a multiple of 45, so the remainder when we divide N by 45 is 9.

[2007 AMC12B #24] How many pairs of positive integers (a, b) are there with $\gcd(a, b) = 1$ and $\frac{a}{b} + \frac{14b}{9a}$ an integer?

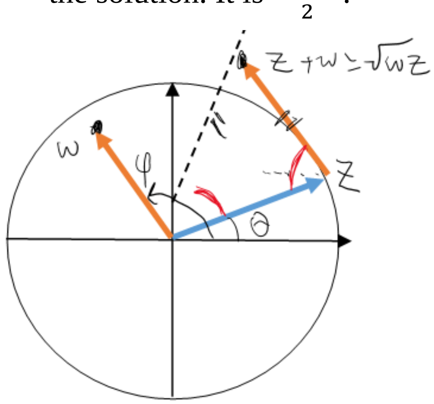
We can begin by adding the fractions: $\frac{9a^2+14b^2}{9ab}$ is an integer so the numerator is a multiple of 9, but that means $14b^2$ is a multiple of 9, which implies $b = 3k$ for some integer k. Substitute that in and reduce the fraction to $\frac{a^2+14k^2}{3ak}$. Next, since the numerator must be a multiple of a, $14k^2$ must also be a multiple of a, but a and k have no common factors, so 14 is a multiple of a, which gives a just 4 possible value. Using the same trick one more time, we see that a^2, and thus a, must also be a multiple of k, but still they have no common factors. This means $k = 1$ and $b = 3$. There are now only 4 pairs to check, and indeed $a = 1, 2, 7$ *or* 14 with $b = 3$ all work.

Complex Numbers

My opinion is that complex numbers are underappreciated and unjustly maligned by the majority of the population. It’s unfortunate that most people see them, only briefly, as solutions to quadratics with no real roots, and don’t learn much more besides how to do some unmotivated arithmetic. Complex numbers are just as real as any other number (i.e. they are all abstractions that are highly useful for modeling the real world). It is too bad that students often don’t learn how to think of complex numbers and their arithmetic from a more geometric viewpoint.

[2018 AMC12B #16] Three solutions to the equation $(z + 6)^8 = 81$ form a triangle in the complex plane. What is the least possible area of that triangle?

The eight solutions form a regular octagon, centered at -6, with distance $\sqrt{3}$ from center to each vertex. From here we can use plane geometry rather than dealing with complex numbers to find the solution. It is $\frac{3\sqrt{2}-3}{2}$.



[2014 AIME II #10] Suppose that complex number z has $|z| = 2014$. There is a polygon in the complex plane with vertices at z and every w such that $\frac{1}{z+w} = \frac{1}{z} + \frac{1}{w}$. Find the area of the polygon.

First, we use some algebra to simplify the equation. Multiply by the common denominator to get $zw = w^2 + 2zw + z^2 = (z + w)^2$. But now we can use a geometric understanding of complex arithmetic to resolve our algebra problem. $z + w$ and a $\sqrt{zw}$ should lie on the bisector of the angle formed by z and w

and have equal length. So, the marked red angles are equal. In the diagram's notation, $\frac{\phi+\theta}{2} - \theta = \theta + (\pi - \phi)$, which simplifies to $\phi - \theta = \frac{2\pi}{3}$. To account for the assumptions of how the diagram was drawn, we should actually consider that $|\phi - \theta| = \frac{2\pi}{3}$. This means that the triangle is in fact equilateral, $|w| = |z| = 2014$. There are two such w, and the polygon described in the problem is a regular triangle inscribed in a circle of radius 2014.